# Summary of Chapter 3

We introduce arithmetic functions, functions from $\mathbb{N}$ to $\mathbb{C}$. To each arithmetic function $f$ we associate a Dirichlet Series, $D_f(s)$. Looking at the Dirichlet product of two such series $D_f(s)\,D_g(s)$ we are formally led to a new Dirichlet series with coefficients given by the Dirichlet convolution $f * g$, thus formally

$$D_{f*g}(s) = D_f(s)\,D_g(s)\,.$$

We prove that the convolution of multiplicative functions is multiplicative.

**Question** How to factor a given arithmetic function $F$?

**Factor the Dirichlet Series**. If $F$ is a multiplicative arithmetic function than the Dirichlet Series $D_F(s)$ has an Euler product. Factor the Euler product of $D_F(s)$ into "simpler" Euler products, which in turn equal Dirichlet Series $D_f(s)$ for various $f$. This "suggests" that $F$ is the convolution of these $f$. Usually these $D_f(s)$ will be of the form $\zeta(\ell s)$ or $\zeta^{-1}(ks)$ for various $k$ and $\ell$. We introduce the arithmetic function $\mu_k$ by

$$\frac{1}{\zeta(ks)} = \sum_{n=1}^{\infty} \frac{\mu_k(n)}{n^s},$$

for $\operatorname{Re} s > 1/k$.

As noted in the lectures, if $f$, $g$ and $F$ are multiplicative then it suffices to prove equality on prime powers. It was also noted that

$$(f * g)(p^r) = \sum_{0 \le k \le r} f\big(p^k\big)\,g\big(p^{r-k}\big) = \sum_{a+b=r} f(p^a)\,g\big(p^b\big)\,.$$

A number of arithmetic functions are introduced and relations are found between them. The most important relation is Möbius Inversion $1 * \mu = \delta$, or equivalently

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$